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A Chebyshev Expansion of Singular Integral Equations with a Logarithmic Kernel

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Singular integral equations of the first kind with a symmetric kernel having a logarithmic singularity are studied. An algorithm based on the expansion in Chebyshev polynomials of the first kind is presented. The method is applied to two-dimensional wave-scattering and convergence is demonstrated.

1. INTRODUCTION

Two-dimensional wave-scattering by thin open scatterers leads to integral equations of the form

$$\int_{-1}^{1} F(t)\{B(s,t)\ln|s-t| + C(s,t)\}\,dt = g(s), \qquad |s|<1, \tag{1.1}$$

where $F(t)$ is the unknown, $g(s)$ is the forcing function, $B(s,t)$ and $C(s,t)$ are symmetric regular kernels ($B(0,0)\neq 0$). Because of the weak singularity of the kernel, Eq. (1.1) has not attracted much attention. MacGamy [1] discussed the analytic properties of the solution $F(t)$. However, his work was restricted to analytic forcing function and kernels. Hayashi [2] studied thoroughly the scattering problem which leads to Eq. (1.1). He suggested converting (1.1) into a Cauchy-type integral equation, and using the well-developed theory of these equations [3].

Following McCamy and Hayashi we assume that the solution $F(t)$ is Hölder-continuous in $(-1,1)$, and $f(t)=(1-t^2)^{1/2}F(t)$ has a continuous extension to $t=\pm 1$. We reformulate equation $(-1,1)$ in the form

$$\int_{-1}^{1} \frac{f(t)}{(1-t^2)^{1/2}}\{\ln|s-t| + D(s,t)\}\,dt = g(s), \qquad |s|<1. \tag{1.2}$$

In this work we study the solution of (1.2) by expansion in Chebyshev polynomials of the first kind. This method has been applied by Gladwell and Coen [4] to the solution of microstrip problems. Recently, Moss and Christensen [5] have studied the special case of scattering by a strip by the same method. They have discussed some

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theoretical aspects concerning the convergence and the stability of the method. Independently, the author has used the same method for wave scattering by a general smooth open obstacle [6].

The present work is essentially a generalization of the one presented by Moss and Christensen [5]. While in the strip problem $D(s, t)$ (1.2) is a function of $|s-t|$ only, this is not true generally for other open scatterers. Therefore, all theoretical and numerical considerations which are based on a convolution-type kernel cannot be utilized.

Although we formulate the theory quite generally, we refer specifically to the scattering of acoustic waves by a hard thin obstacle; or equivalently, the scattering of an E-polarized electromagnetic wave by a thin conductor [2, 6]. Let $\boldsymbol{\rho}(s) = (x(s), y(s))$, $|s| \leqslant 1$ define the scatterer, then the kernel of our integral equation is $H_0^{(2)}(k\,|\boldsymbol{\rho}(s)-\boldsymbol{\rho}(t)|)$, where k is the wave number. However,

$$\begin{aligned}(j\pi/2)\, H_0^{(2)}(z) = \ln z + \{J_0(z) - 1\} \ln z - Q(z)\\ + (j\pi/2)\{1 - (2j/\pi)(\gamma - \ln 2)\}\, J_0(z),\end{aligned} \tag{1.3}$$

where γ is the Euler constant and

$$Q(z) = \sum_{m=1}^{\infty} \frac{(-)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right). \tag{1.4}$$

Comparing (1.3) with (1.2) we find

$$\begin{aligned}D(s, t) = \ln((k\,|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|)/|s-t|) - Q(k\,|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|)\\ + \{J_0(|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|) - 1\} \ln |\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|\\ - (j\pi/2)\{1 - (2j/\pi)(\gamma - \ln 2)\}\, J_0(|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|).\end{aligned} \tag{1.5}$$

We assume in addition that

$$U(s, t) = k\,|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|/|s-t| \tag{1.6}$$

is differentiable in the square $|s|$, $|t| \leqslant 1$. So that

$$D(s, t) = (s-t)^2\, L(s, t) \ln |s-t| + M(s, t), \tag{1.7}$$

with $L(s, t)$ and $M(s, t)$ bounded differentiable functions in the square $|s|$, $|t| \leqslant 1$.

With these kinds of kernels in mind we proceed to develop systematically the theory for the solution of Eq. (1.2). We further discuss the numerical details of the method, and present several numerical examples.

2. Theory

The form of Eq. (1.2) naturally suggests that we look for our solution $f(t)$ in the Hilbert space $L^2(\Gamma, w)$ of all functions square integrable on $\Gamma = (-1, 1)$ with respect to the weight $w(x) = (1-x^2)^{-1/2}$. It is well known that the Chebyshev polynomials of the first kind $\{T_n(x)\}$ comprise a complete orthogonal set in $L^2(\Gamma, w)$. Moreover, the series $\sum_{n=0}^{\infty} \gamma_n f_n T_n(x)$ ($\gamma_0 = 0.5$, $\gamma_n = 1$, $n > 0$) with

$$f_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\, T_n(x)}{(1-x^2)^{1/2}}\, dx, \tag{2.1}$$

converges in the mean to the function $f(x)$. The norm in this Hilbert space is simply

$$\|f\|^2 = \int_{-1}^{1} \frac{|f(x)|^2}{(1-x^2)^{1/2}}\, dx = \frac{\pi}{2} \sum_{n=0}^{\infty} \gamma_n |f_n|^2. \tag{2.2}$$

We are specifically interested in integral operators within $L^2(\Gamma, w)$ of the form

$$\mathscr{K} f = \int_{-1}^{1} \frac{K(s,t) f(t)}{(1-t^2)^{1/2}}\, dt. \tag{2.3}$$

We restrict the discussion to compact completely continuous operators which obey the sufficient condition

$$\|\mathscr{K}\|^2 = \int_{-1}^{1} \int_{-1}^{1} \frac{|K(s,t)|^2\, ds\, dt}{(1-s^2)^{1/2}(1-t^2)^{1/2}} < \infty. \tag{2.4}$$

We represent these operators by matrices, using the Chebyshev polynomials as our basis functions. The most significant property of these matrices is that they are almost finite. This can be stated in

THEOREM 1. *Let $\mathscr{K}$ (Eq. (2.3)) be an integral operator that obeys condition* (2.4), *then the double expansion*

$$K_{MN}(s,t) = \sum_{m=0}^{M} \sum_{n=0}^{N} \gamma_m \gamma_n K_{mn} T_m(s)\, T_n(t) \tag{2.5}$$

with

$$K_{mn} = \frac{4}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{K(s,t)\, T_m(s)\, T_n(t)}{(1-s^2)^{1/2}(1-t^2)^{1/2}}\, ds\, dt \tag{2.6}$$

converges in the mean to $K(s,t)$ as $M, N \to \infty$. That is, for any $\varepsilon > 0$ there are M_0 and N_0 such that for every $M > M_0$ and $N > N_0$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{|K(s,t) - K_{MN}(s,t)|^2}{(1-s^2)^{1/2}(1-t^2)^{1/2}}\, ds\, dt < \varepsilon. \tag{2.7}$$

Also

$$\| \mathscr{K} \|^2 = \frac{\pi^2}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_m \gamma_n |K_{mn}|^2 < \infty. \tag{2.8}$$

We study now the special operators (1.2), (1.5) under discussion. We state the following result proven by Moss and Christensen [5]:

THEOREM II. *The integral operator* $\mathscr{A}_0$

$$\mathscr{A}_0 f = \int_{-1}^{1} \frac{\ln |s-t| f(t)}{(1-t^2)^{1/2}}\, dt, \qquad f \in L^2(\Gamma, w) \tag{2.9}$$

is a diagonal completely continuous operator. That is,

$$\mathscr{A}_0 T_n(t) = \int_{-1}^{1} \frac{\ln |s-t|\, T_n(t)}{(1-t^2)^{1/2}}\, dt = -\pi v_n T_n(s) \tag{2.10}$$

with $v_0 = \ln 2$, $v_n = 1/n$, $n > 0$.

Equivalently,

$$\ln |s-t| \sim -2 \sum_{n=0}^{\infty} \gamma_n v_n T_n(s)\, T_n(t). \tag{2.11}$$

Clearly,

$$\| \mathscr{A}_0 \|^2 = \pi^2 \sum_{n=0}^{\infty} v_n^2 = \pi^2 \left(\ln^2 2 + \frac{\pi^2}{6} \right). \tag{2.12}$$

We find that if $f \in L^2(\Gamma, w)$

$$\mathscr{A}_0 f \sim -\pi \sum_{n=0}^{\infty} \gamma_n v_n f_n T_n(s), \tag{2.13}$$

so that the range $R(\mathscr{A}_0)$ *is the subspace of all functions* $g(s)$ *for which* $\sum_{n=1}^{\infty} n^2 |g_n|^2 < \infty$.

We extend Theorem II following the form (1.7) of the regular part of the kernel $D(s,t)$.

THEOREM III. *The integral operator* $\mathscr{H}$

$$\mathscr{H} f = \int_{-1}^{1} \frac{(s-t)^2 \ln |s-t|\, L(s,t)}{(1-t^2)^{1/2}} f(t)\, dt \tag{2.14}$$

obeys (2.4) *if* $L(s,t)$ *is bounded in the square* $|s|, |t| \leqslant 1$.

This is obvious since

$$\|\mathscr{W}\|^2 = \int_{-1}^{1} \frac{(s-t)^4 \,|\ln|s-t||^2\, |L(s,t)|^2}{(1-t^2)}\, ds \leqslant |16\,|L_b||^2\, \|\mathscr{A}_0\|^2 < \infty, \tag{2.15}$$

where L_b is the bound of $L(s,t)$. Therefore $D(s,t)$ (1.7) is a completely continuous operator that can be expanded in Chebyshev polynomials.

As a simple example consider the case of $L(s,t) = constant$. We can easily prove that the operator $\mathscr{T}$

$$\mathscr{T}f = \int_{-1}^{1} \frac{(s-t)^2 \ln|s-t|}{(1-t^2)^{1/2}}\, f(t)\, dt, \tag{2.16}$$

is represented by a matrix J whose elements are

$$J_{nn} = \gamma_{n+1} v_{n+1} + \gamma_{n-1} v_{n-1} - 2\gamma_n v_n, \qquad J_{n,n+2} = J_{n+2,n} = -\tfrac{1}{2} J_{n+1,n+1}. \tag{2.17}$$

For $n > 1$ we can simplify (2.17) *to*

$$J_{nn} = \frac{2}{n(n^2-1)}, \qquad J_{n,n+2} = J_{n+2,n} = -\frac{1}{n(n+1)(n+2)}. \tag{2.18}$$

We can show that for a smooth enough function $L(s,t)$ the matrix elements W_{mn} decay at least as $n^{-3}(m^{-3})$ for $n \gg 1$ $(m \gg 1)$.

We shall see that this difference between the matrix elements of $\mathscr{A}_0$ and those of $\mathscr{W}$ leads directly to a simple regularization scheme for the solution of our integral equation (1.2).

It is clear from Theorem I that the integral equation $\mathscr{A}_0 f = g$ has a solution in $L^2(\Gamma, w)$ if $g \in L^2(\Gamma, w)$ and $\sum_{n=1}^{\infty} n^2 |g_n|^2 < \infty$, where g_n are defined by (2.1). The solution can be presented in the form

$$f(t) \sim -\frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\gamma_n}{v_n}\, g_n T_n(t). \tag{2.19}$$

Equation (2.19) defines essentially a transformation $\mathscr{S}: R(\mathscr{A}_0) \to L^2(\Gamma, w)$ such that $f = Sg$ [2].

We turn now to the general case (1.2). Define

$$\mathscr{D}f = \int_{-1}^{1} \frac{D(s,t) f(t)}{(1-t^2)^{1/2}}\, dt \tag{2.20}$$

and

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{D}. \tag{2.21}$$

We state now

THEOREM IV. *If $\mathscr{S}\mathscr{D}$ is compact and the null space of $\mathscr{I} + \mathscr{S}\mathscr{D}$ is trivial, the equation $\mathscr{A}f = g$ with $g \in R(\mathscr{A}_0)$ has a unique solution $f = (\mathscr{I} + \mathscr{S}\mathscr{D})^{-1} \mathscr{S}_g$.*

Proof. Since the null space of $\mathscr{S}$ is trivial, the equation $(\mathscr{A}_0 + \mathscr{D})f = g$ is equivalent to $(\mathscr{I} + \mathscr{S}\mathscr{D})f = \mathscr{S}_g$. Clearly $\mathscr{S}\mathscr{D}$ is bounded, and therefore $R(\mathscr{A}) \subseteq R(\mathscr{A}_0)$. The existence of an inverse $(\mathscr{I} + \mathscr{S}\mathscr{D})^{-1}$ follows from the well-known Fredholm theorems [7].

The elements of the operator $\mathscr{S}\mathscr{D}$ are simply given by

$$(\mathscr{S}\mathscr{D})_{0n} = 2 \ln 2D_{0n}, \qquad (\mathscr{S}\mathscr{D})_{mn} = \tfrac{1}{2} m D_{mn}. \tag{2.22}$$

We finally prove a sufficient condition for $\mathscr{S}\mathscr{D}$ to be a compact completely continuous operator.

THEOREM V. *$\mathscr{S}\mathscr{D}$ is a completely continuous operator if the operator $\mathscr{K}$,*

$$\mathscr{K}f = \int_{-1}^{1} \frac{\partial D/\partial s}{(1-t^2)^{1/2}} f(t)\, dt \tag{2.23}$$

obeys condition (2.4).

Proof. Since $\mathscr{D}$ is a completely continuous operator

$$D(s, t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_m \gamma_n D_{mn} T_m(s)\, T_n(t). \tag{2.24}$$

Therefore,

$$\frac{\partial D(s, t)}{\partial s} \sim \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \gamma_n m D_{mn} U_{m-1}(s)\, T_n(t), \tag{2.25}$$

where $U_m(s)$ are the Chebyshev polynomials of the second kind. Equivalently,

$$\begin{aligned} mD_{mn} &= \left(\frac{2}{\pi}\right)^2 \int_{-1}^{1} \int_{-1}^{1} \frac{\partial D}{\partial S} \left(\frac{1-s^2}{1-t^2}\right)^{1/2} U_{m-1}(s)\, T_n(t)\, ds\, dt \\ &= \left(\frac{2}{\pi}\right)^2 \int_{-1}^{1} \int_{-1}^{1} \frac{\partial D}{\partial S} \frac{0.5}{(1-t^2)^{1/2}(1-s^2)^{1/2}} [T_{m-1}(s) - T_{m+1}(s)]\, T_n(t)\, ds\, dt \\ &= \frac{1}{2} (H_{m-1,n} - H_{m+1,n}) \qquad m \geqslant 1, \quad n \geqslant 0. \end{aligned} \tag{2.26}$$

Therefore,

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 |D_{mn}|^2 &= \tfrac{1}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |H_{m-1,n} - H_{m+1,n}|^2 \\ &\leqslant \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{|H_{m-1,n}|^2 + |H_{m+1,n}|^2\} < \infty. \end{aligned}$$

3. Numerical Results

An algorithm based on the previous theory has been written for the solution of two-dimensional scattering problems [6]. The expansion in Chebyshev polynomials has been accomplished by the Gauss–Chebyshev quadrature. That is,

$$g_m = \frac{2}{\pi}\int_{-1}^{1} \frac{g(s)\,T_m(s)}{(1-s^2)^{1/2}} \simeq \left(\frac{2}{M+1}\right)\sum_{i=0}^{M} g(x_i)\,T_m(x_i) \tag{3.1}$$

and

$$D_{mn} = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{D(s,t)\,T_m(s)\,T_n(t)}{(1-s^2)^{1/2}(1-t^2)^{1/2}}\,ds\,dt$$
$$\simeq \left(\frac{2}{M+1}\right)^2 \sum_{i,j=0}^{M} D(x_i,x_j)\,T_m(x_i)\,T_n(x_j),$$

where

$$x_i = \cos\left[\frac{(2i+1)\pi}{2M+2}\right]. \tag{3.3}$$

We construct our solution according to Theorem IV, using in addition Eqs. (2.22) and (2.19).

Obviously, all infinite representations are approximated by finite ones (Eq. (2.5)). The finite dimension N is increased until convergence is achieved. We note that because of the expansion in orthogonal polynomials any $N \times N$ matrix is a submatrix of a larger $(N+N') \times (N+N')$ matrix. Thus, only the additional elements of the larger matrix have to be calculated. Also, the inversion of an $(N+N') \times (N+N')$ matrix can be accomplished by partitioning, using the known inverse of the $N \times N$ matrix.

In the sequel we present the solution of integral equations (1.2) with kernels given by Eq. (1.5) for different scatterers $\boldsymbol{\rho}(s)$. The first example is scattering by a strip $\boldsymbol{\rho}(s) = (ws, 0)$. The integral equation is

$$\int_{-1}^{1} \frac{f(s)}{(1-s^2)^{1/2}} H_0^{(2)}(kw\,|s-t|)\,ds = e^{-jkwt\cos\alpha}. \tag{3.4}$$

The parameters chosen for this example are $\alpha = \pi/2$, $kw = 3\pi$. In Table I we present the expansion coefficients f_n of the unknown function $f(s)$ for $N = 6$, 10, 14, respectively. Convergence is fast, and for all practical applications the $N = 10$ accuracy is sufficient.

The second example is scattering by a semicircular cylinder, $\boldsymbol{\rho}(s) = (a\cos(\pi t/2)$, $a\sin(\pi t/2))$. The integral equation under consideration is

$$\int_{-1}^{1} \frac{f(s)}{(1-s^2)^{1/2}} H_0^{(2)}\left[2ka\sin\left(\frac{\pi}{4}|s-t|\right)\right] ds = e^{jka\cos(\alpha-\pi t/2)}. \tag{3.5}$$

TABLE I

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Strip ($kw = 3\pi$)

n	$N = 6$	$N = 10$	$N = 14$
0	$3.0002 - j.23184$	$2.9988 - j.22934$	$2.9988 - j.22933$
2	$-1.9622 - j.23403$	$-1.9619 - j.22652$	$-1.9618 - j.22651$
4	$-0.29185 - j.31923$	$-0.27424 - j.30992$	$-0.27422 - j.30994$
6	$-0.00969 - j.10388$	$0.04900 - j.15110$	$0.04898 - j.15115$
8		$0.07902 + j.05033$	$0.07895 + j.05042$
10		$-0.03445 + j.00032$	$-0.03346 - j.00003$
12			$0.00646 - j.00207$
14			$-0.00072 + j.00045$

TABLE II

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Semicircular Strip ($ka = \pi$)

n	$N = 8$	$N = 12$	$N = 16$
0	$-0.39469 + j.33896$	$-0.39468 + j.33897$	$-0.39468 + j.33897$
1	$-0.25970 + j.50436$	$-0.25971 + j.50428$	$-0.25971 + j.50428$
2	$-0.35534 + j.15328$	$-0.35532 + j.15329$	$-0.35532 + j.15329$
3	$-0.07722 - j.47570$	$-0.07743 - j.47572$	$-0.07743 - j.47572$
4	$0.42156 - j.22968$	$0.42160 - j.22973$	$0.42160 - j.22973$
5	$0.40825 + j.23431$	$0.40796 + j.23488$	$0.40796 + j.23488$
6	$-0.13362 + j.33987$	$-0.13376 + j.33971$	$-0.13376 + j.33971$
7	$-0.25473 - j.05907$	$-0.24840 - j.05913$	$-0.24840 - j.05913$
8	$0.02437 - j.17999$	$0.02436 - j.17725$	$0.02436 - j.17725$
9		$0.10604 + j.00960$	$0.10603 + j.00960$
10		$-0.00291 + j.06149$	$-0.00291 + j.06149$
11		$-0.03368 - j.00128$	$-0.03348 - j.00126$
12		$0.00027 - j.01735$	$0.00027 - j.01726$
13			$0.00856 + j.00010$
14			$-0.00001 + j.00409$
15			$-0.00189 - j.00001$
16			$0.00001 - j.00085$

Here a is the radius of the circle chosen such that $ka = \pi$. A nonsymmetric excitation is studied ($\alpha = \pi/2$). In Table II we summarize the expansion coefficients for $N = 8$, 12, 16, respectively. Here the odd modes are also excited. Again good convergence is demonstrated.

Our final example is the scattering by a parabolic reflector, $\mathbf{p}(s) = (ws, wqs^2)$. The integral equation to be solved is

$$\int_{-1}^{1} \frac{f(s)}{(1-s^2)^{1/2}} H_0^{(2)}[kw\,|s-t|\,\{1+q^2(s+t)^2\}^{1/2}]\, ds = e^{jkwt(\cos\alpha + qt\sin\alpha)}. \tag{3.6}$$

TABLE III

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Parabolic Reflector ($q = 1$, $kw = 2\pi$)

n	$N = 8$	$N = 12$	$N = 16$
0	$0.40624 - j.23248$	$0.40701 - j.23465$	$0.40701 - j.23466$
2	$-1.8807 - j2.0152$	$-1.8844 - j2.0154$	$-1.8844 - j2.0154$
4	$2.4033 + j1.1770$	$2.4043 + j1.1770$	$2.4043 + j1.1770$
6	$1.0739 + j.35754$	$1.0613 + j.36836$	$1.0613 + j.36838$
8	$-0.37647 - j.31814$	$-0.38720 - j.30761$	$-0.38721 - j.30758$
10		$-0.12618 + j.06888$	$-0.12582 + j.06842$
12		$0.00535 + j.02002$	$0.00573 + j.01956$
14			$0.00829 - j.01047$
16			$0.00102 - j.00058$

A symmetric excitation is considered ($\alpha = \pi/2$) together with the parameters $q = 1$, $kw = 2\pi$. In this case the kernel does not depend on $|s - t|$ only. In Table III we present the expansion coefficients for $N = 8$, 12, 16, respectively. Convergence is again very fast, and rather small matrices must be used.

References

1. R. C. MacCamy, *J. Math. Mech.* (1958), 355.
2. Y. Hayashi, *J. Math. Anal. Appl.* **44** (1973), 489.
3. V. V. Ivanov, "The Theory of Approximate Methods and Their Application to the Numerical Solution of Singular Integral Equations," Noordhoff, Groningen, 1976.
4. G. M. L. Gladwell and S. Coen, *IEEE Trans. Microwave Theory Tech.* **MTT-23** (1975), 805.
5. W. F. Moss and M. J. Christensen, *J. Integ. Equations* **4** (1980), 299.
6. A. Frenkel, External modes of two-dimensional thin scatterers, *IEE, Part H* **130** (1983), 209.
7. W. Schmeudler, "Linear Operators in Hilbert Space," Academic Press, New York, 1965.